

Lecture 2f:

①

Riemann Integration:

We are going to define the integral of certain functions (which we will eventually call Riemann integrable) in a way that will clearly represent the area under the graph of f .

We will denote the integral of f by $\int f$.

- Recall that bounded intervals in \mathbb{R} are any intervals of the form $[a, b]$, $[a, b)$, $(a, b]$, (a, b) , for $a, b \in \mathbb{R}$.
(we also consider every singleton $\{a\} = [a, a]$ an interval).
- For each $E \subseteq \mathbb{R}$, we denote by χ_E the characteristic function of E , i.e. the function $\chi_E: \mathbb{R} \rightarrow \mathbb{R}$,

with

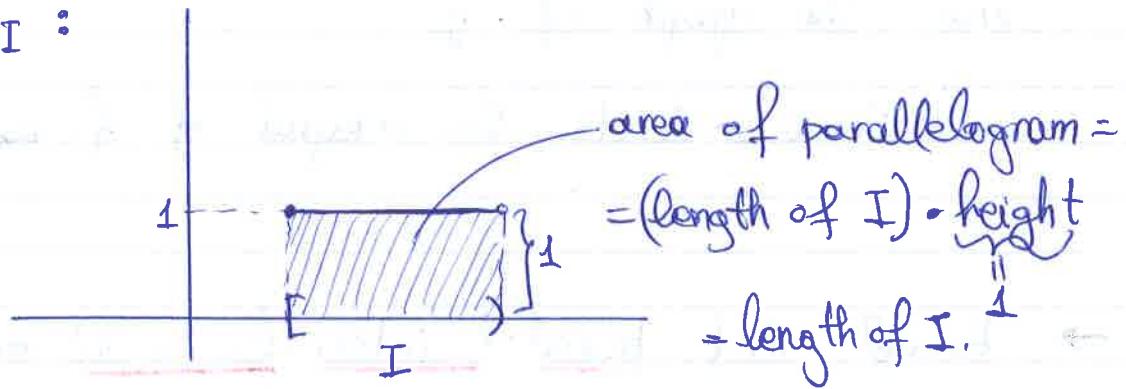
$$\chi_E(x) = \begin{cases} 1, & \text{if } x \in E \\ 0, & \text{otherwise} \end{cases}$$

(2)

→ Def: for any bounded interval $I \subseteq \mathbb{R}$, we define

$$\int \chi_I := \text{length of } I.$$

Notice that this truly represents the area under the graph of χ_I :



We wish to extend this definition to a wider class of functions:

→ Def: We say that $\phi: \mathbb{R} \rightarrow \mathbb{R}$ is a step function if there exist real numbers $x_0 < x_1 < \dots < x_n$ (for some $n \in \mathbb{N}$), such that

- $\phi(x) = 0$ for $x < x_0$ and $x > x_n$, and
- ϕ is constant on (x_{i-1}, x_i) , for all $1 \leq i \leq n$.

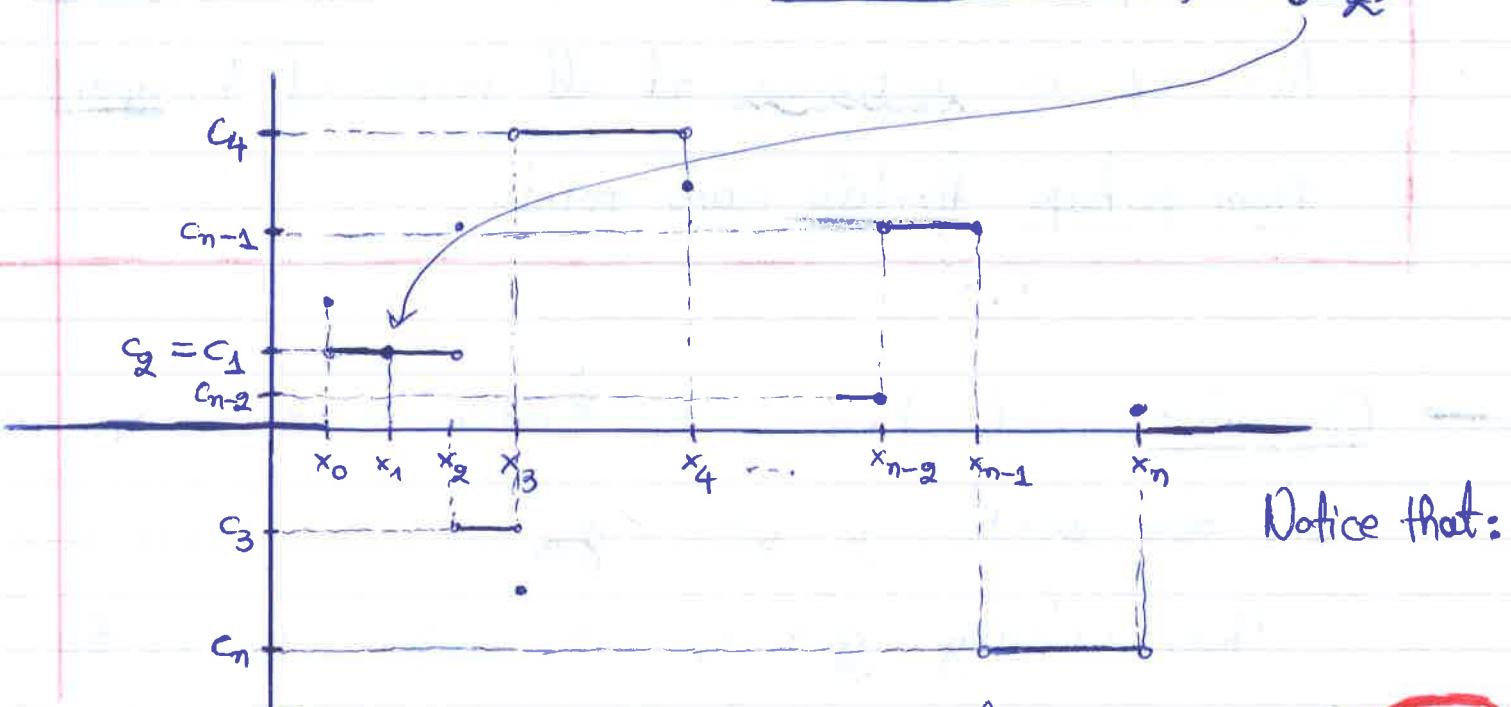
(3)

(Notice that the domain of ϕ is \mathbb{R} , thus ϕ is also defined at x_0, x_1, \dots, x_n ; it's just that, no matter what those values are, as long as the two bullet points above hold, we still call ϕ a step function.)

We say that the ϕ above is a step function with respect to $\{x_0, x_1, x_2, \dots, x_n\}$.

⚠ It's not necessarily true that f changes value as it passes from (x_0, x_1) to (x_1, x_2) !

It could be constant on the whole of (x_0, x_2) .

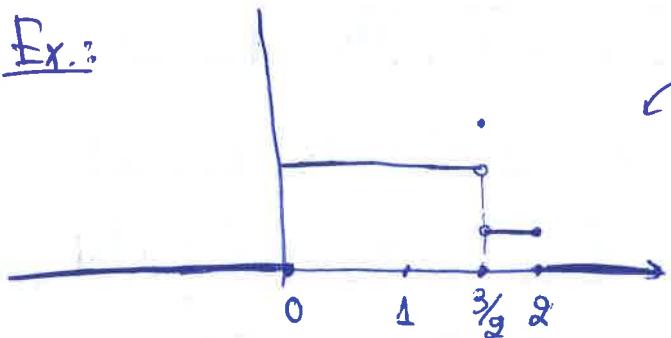


$f: \mathbb{R} \rightarrow \mathbb{R}$ is a step function w.r.t. $\{x_0, x_1, x_2, \dots, x_n\}$ \iff

there exist constants c_1, c_2, \dots, c_n , s.t. $\phi(x) = \sum_{i=1}^n c_i \chi_{(x_{i-1}, x_i)}(x)$,
for all $x \in \mathbb{R} \setminus \{x_0, x_1, \dots, x_n\}$.

(4)

→ Ex.:



A step function

w.r.t. 0, 1, $\frac{3}{2}$, 2.Also, it is a step function
w.r.t. 0, $\frac{3}{2}$, 2.→ Observation :

Let $\phi : \mathbb{R} \rightarrow \mathbb{R}$ be a step function. Then, ϕ is bounded, and there exists an interval $I \subseteq \mathbb{R}$ s.t. $\phi(x) = 0 \ \forall x \notin I$ (i.e., ϕ has bounded support.)

Also, ϕ is continuous at all points of \mathbb{R} , apart from perhaps finitely many points.

→ Observation: Let ϕ be a step function w.r.t. $x_0 < x_1 < \dots < x_n$, and also w.r.t. $y_0 < y_1 < \dots < y_m$. Then, ordering ~~$x_0, x_1, \dots, x_n, y_0, y_1, \dots, y_m$~~ on the axis, and renaming them as $z_0, z_1, z_2, \dots, z_k$, then ϕ is a step function w.r.t. $z_0, z_1, z_2, \dots, z_k$.

(5)

→ Def: Let ϕ be a step function w.r.t. $\{x_0, \dots, x_n\}$,
 where $x_0 < x_1 < \dots < x_n$, taking the value c_i
 on (x_{i-1}, x_i) , $\forall i=1, \dots, n$.

$$\text{i.e. : } \phi(x) = \sum_{i=1}^n c_i \chi_{(x_{i-1}, x_i)}(x), \quad \forall x \neq x_0, x_1, \dots, x_n.$$

We define $\int \phi = \sum_{i=1}^n c_i \cdot \int \chi_{(x_{i-1}, x_i)}$

$$= \left[\sum_{i=1}^n c_i \cdot (x_{i-1}, x_i) \right].$$



We need to make sure that the above is a good definition, i.e. that different $\{x_0, \dots, x_n\}$ w.r.t. which ϕ is a step function lead to the same integral.

For this, we will use the following:

→ Obs.: Let ϕ be a step function w.r.t. $\{x_0, x_1, \dots, x_n\}$.

Let $\{y_0, \dots, y_m\}$ be a finite subset of \mathbb{R} .

Then, ϕ is a step function w.r.t. $\{x_0, x_1, \dots, x_n\} \cup \{y_0, \dots, y_m\}$.

Proof: It suffices to show that ϕ is a step function w.r.t. $\{x_0, x_1, \dots, x_n\} \cup \{y\}$, for any $y \in \mathbb{R}$.

(6)

Then we repeat this process of adding an extra element to $\{x_0, x_1, \dots, x_n\}$ m times, once for each y_i .

Let $y \in \mathbb{R}$, $y \neq x_0, x_1, \dots, x_n$.

We assume that $x_j < y < x_i$, for some $j \in \{1, \dots, n\}$.

(and we work similarly when $y < x_0$ or $y > x_n$).

Since ϕ is a step function w.r.t. $\{x_0, x_1, \dots, x_n\}$,

we have that $\exists c_i \in \mathbb{R}$, for all $i = 1, \dots, m$,

$$\text{s.t. } \phi(x) = \sum_{i=1}^m c_i \cdot \chi_{(x_{i-1}, x_i)}(x), \forall x \neq x_0, x_1, \dots, x_n.$$

$$\text{Thus, } \phi(x) = \sum_{i \in \{1, \dots, m\} \setminus \{j\}} c_i \cdot \chi_{(x_{i-1}, x_i)}(x) +$$

$$+ c_j \cdot \chi_{(x_{j-1}, x_j)}(x) \quad \xrightarrow{(x_{j-1}, x_j) = (x_{j-1}, y] \cup [y, x_j)}$$

$$\begin{aligned} \phi(x) &= \sum_{i \in \{1, \dots, m\} \setminus \{j\}} c_i \cdot \chi_{(x_{i-1}, x_i)}(x) + c_j \cdot \chi_{(x_{j-1}, y)}(x) + \\ &\quad + c_j \cdot \chi_{(y, x_j)}(x), \forall x \neq x_0, \dots, x_n, y. \end{aligned}$$

so, ϕ is a step function w.r.t. $\{x_0, x_1, \dots, x_n\} \cup \{y\}$.

(7)

→ for ϕ a step function, $\int \phi$ above is well defined:

Proof: Let ϕ be a step function

w.r.t. $\{x_0, \dots, x_n\}$
and w.r.t. $\{y_0, \dots, y_m\}$.

Then, $\{x_0, \dots, x_n\}$
and $\{y_0, \dots, y_m\}$ lead
to the same integral
for ϕ :

We know that ϕ is a step function w.r.t.
 $\{x_0, \dots, x_n\} \cup \{y_0\}$, and, by $\textcircled{*}$ in previous
page, $\{x_0, \dots, x_n\} \cup \{y_0\}$ and $\{x_0, \dots, x_n\}$ lead
to the same integral of ϕ .

By induction, we get that $\{x_0, \dots, x_n\}$ and
 $\{x_0, \dots, x_n\} \cup \{y_0, \dots, y_m\}$ lead to the same integral for ϕ .

Similarly, $\{y_0, \dots, y_m\}$ and $\{x_0, \dots, x_n\} \cup \{y_0, \dots, y_m\}$
lead to the same integral for ϕ .

Thus, $\{y_0, \dots, y_m\}$ and $\{x_0, \dots, x_n\}$ lead to the
same integral for ϕ (as $\{x_0, \dots, x_n\} \cup \{y_0, \dots, y_m\}$). ■

⑧

(Linearity of integral for step functions).

→ Prop: Let ϕ, ψ be step functions

and $\lambda, \mu \in \mathbb{R}$. Then, $\lambda\phi + \mu\psi$ is a step function,

$$\text{and } \int(\lambda\phi + \mu\psi) = \lambda\int\phi + \mu\int\psi.$$

Proof: We list all the potential jump points

$x_0 < x_1 < \dots < x_n$ of ϕ and ψ together.

$$\text{If } \phi(x) = \sum_{i=1}^n c_i \chi_{(x_{i-1}, x_i)}(x) \quad \forall x \neq x_0, \dots, x_n$$

$$\text{and } \psi(x) = \sum_{i=1}^n d_i \chi_{(x_{i-1}, x_i)}(x) \quad \forall x \neq x_0, \dots, x_n,$$

$$\text{then } \lambda\phi(x) + \mu\psi(x) = \sum_{i=1}^n (\lambda c_i + \mu d_i) \chi_{(x_{i-1}, x_i)}(x) \\ \forall x \neq x_0, \dots, x_n, \text{ a step function,}$$

$$\text{thus } \int \lambda\phi + \mu\psi = \sum_{i=1}^n (\lambda c_i + \mu d_i) \cdot (x_i - x_{i-1}) =$$

$$= \lambda \sum_{i=1}^n c_i \cdot (x_i - x_{i-1}) + \mu \sum_{i=1}^n d_i \cdot (x_i - x_{i-1}) =$$

$$= \lambda \int \phi + \mu \int \psi.$$

→ Prop: Let ϕ, ψ be step functions, with

$$\phi(x) \geq \psi(x), \forall x \in \mathbb{R}$$

(or, for all $x \in \mathbb{R}$ apart from the potential jump points of ϕ and ψ).

Then, $\int \phi \geq \int \psi$.

Proof: Exercise. (List all the potential jump points of ϕ and ψ in order, etc).



Def: Let $f: \mathbb{R} \rightarrow \mathbb{R}$. We say that

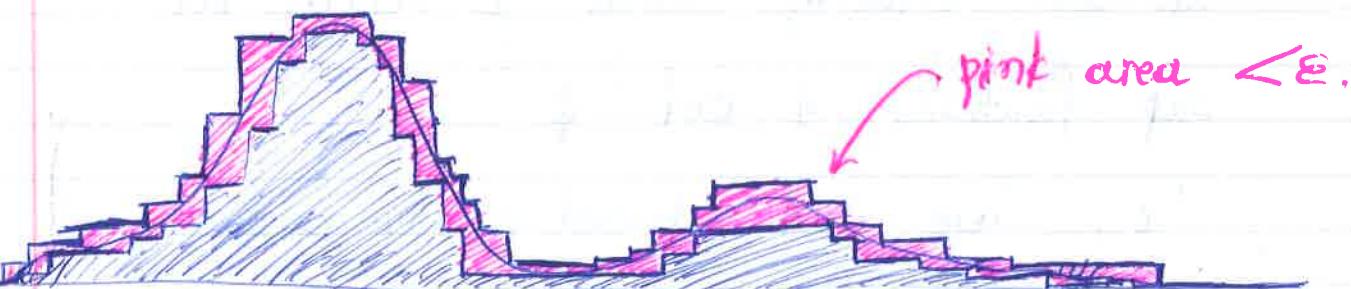
f is Riemann integrable if, $\forall \epsilon > 0$,

there exist step functions ϕ and ψ , such that

$$\phi \leq f \leq \psi$$

$$\text{and } \int \psi - \int \phi < \epsilon.$$

(Δ Notice that, for such ϕ, ψ , $0 \leq \int \psi - \int \phi < \epsilon$.)



\rightarrow Prop: Step functions are Riemann integrable.

Proof: Let ϕ be a step function. Let $\epsilon > 0$.

Then,

$$\underbrace{\phi}_{\text{step}} \leq \phi \leq \underbrace{\psi}_{\text{step}},$$

with

$$\int |\phi - \psi| = 0 < \epsilon.$$

Since $\epsilon > 0$ was arbitrary, ϕ is Riemann integrable. ■

\rightarrow Prop: If $f: \mathbb{R} \rightarrow \mathbb{R}$ is Riemann integrable,

then:

- (i) f is bounded,
- and (ii) f has bounded support.

Proof:

Since f is Riemann integrable,

we can certainly bound f between two step functions, ϕ and ψ .

I.e., there exist ϕ and ψ step functions,

s.t. $\star \quad \phi(x) \leq f(x) \leq \psi(x), \quad \forall x \in \mathbb{R}.$

ϕ and ψ bounded, so f is bounded. Also:



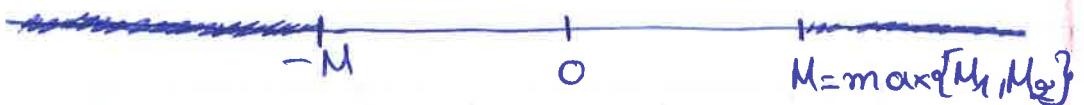
There exists $M_1 > 0$ st. $\psi(x) = 0$ whenever $x \notin [-M_1, M_1]$.

Also, there exists $M_2 > 0$ st. $\phi(x) = 0$ whenever $x \notin [-M_2, M_2]$.

Thus, $\phi(x) = \psi(x) = 0$ whenever $x \notin [-M, M]$,

$$\text{where } M = \max\{M_1, M_2\},$$

therefore $\# \Rightarrow f(x) = 0 \quad \forall x \notin [-M, M]$.



In other words, f has bounded support.



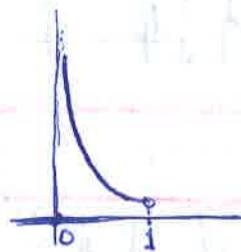
Examples:

(1)

$f: (0, 1) \rightarrow \mathbb{R}$,

$$\text{with } f(x) = \frac{1}{1-x} \quad \forall x \in (0, 1),$$

is not Riemann integrable



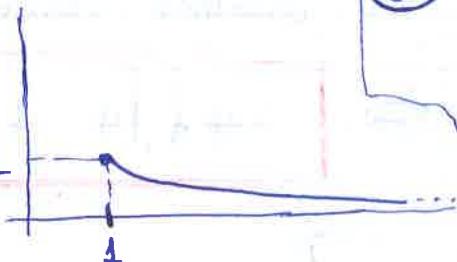
(it is unbounded).

(2)

$f: [1, +\infty) \rightarrow \mathbb{R}$,

$$\text{with } f(x) = \frac{1}{x^2} \quad \forall x \in [1, +\infty),$$

is not Riemann integrable



(3)

is Riemann integrable,
because
it is a
step
function.

(bounded, yet has unbounded support).

(*) And yet we are used to calculating $\int_1^{+\infty} \frac{1}{x^2} dx$! Really, we are calculating $\lim_{M \rightarrow \infty} \int_1^M \frac{1}{x^2} dx$, and $x^2: [1, M] \rightarrow \mathbb{R}$ is Riemann integrable.

Actually, when we take a step function (such as 0) and change it at finitely many pts, the result is Riemann integrable. When we change it at infinitely many pts, anything can happen!

4) $\chi_{\{\frac{1}{n} : n \in \mathbb{N}\}}$ is Riemann integrable. (we will see proof next time. Try it!)



One could say that $\chi_{\{\frac{1}{n}\}}$ is a step function w.r.t. $\{1, \frac{1}{2}\}, \dots, \chi_{\{1, \frac{1}{2}, \dots, \frac{1}{n}\}}$ step w.r.t. $\{1, \dots, \frac{1}{n}\}$, so, $\forall n \in \mathbb{N}, \chi_{\{1, \frac{1}{2}, \dots, \frac{1}{n}\}}$ R-integrable. This doesn't imply that $\chi_{\{\frac{1}{n} : n \in \mathbb{N}\}}$ \mathbb{R} -integrable.

→ Theorem: A function $f: \mathbb{R} \rightarrow \mathbb{R}$ is Riemann integrable



$$\sup \left\{ \int \phi : \phi \text{ is a step function and } \phi \leq f \right\} =$$

$$\inf \left\{ \int \psi : \psi \text{ is a step function and } \psi \geq f \right\}.$$

→ Def: Let $f: \mathbb{R} \rightarrow \mathbb{R}$ Riemann integrable.

We define the integral $\boxed{\int f}$ of f as the common value

$$\boxed{\sup \left\{ \int \phi : \phi \text{ is a step function and } \phi \leq f \right\}}$$



$$\inf \left\{ \int \psi : \psi \text{ is a step function and } \psi \geq f \right\}$$

Actually, when we take a step function (such as 0) and change it at finitely many pts, the result is Riemann integrable. When we change it at infinitely many pts, anything can happen!

4) $\chi_{\{\frac{1}{n} : n \in \mathbb{N}\}}$ is Riemann integrable. (we will see proof next time. Try it!)



One could say that $\chi_{\{1\}}$ is a step function w.r.t. $\{1\}$.
 $\chi_{\{1, \frac{1}{2}\}}$ is step w.r.t. $\{1, \frac{1}{2}\}, \dots, \chi_{\{1, \frac{1}{2}, \dots, \frac{1}{n}\}}$ step w.r.t. $\{1, \dots, \frac{1}{n}\}$
so, $\forall n \in \mathbb{N}, \chi_{\{1, \frac{1}{2}, \dots, \frac{1}{n}\}}$ R-integrable. This doesn't imply that $\chi_{\{\frac{1}{n} : n \in \mathbb{N}\}}$ R-integrable.

→ Theorem: A function $f: \mathbb{R} \rightarrow \mathbb{R}$ is Riemann integrable \Leftrightarrow bounded

$$\sup \left\{ \int \phi : \phi \text{ is a step function and } \phi \leq f \right\}$$

=

$$\inf \left\{ \int \psi : \psi \text{ is a step function and } \psi \geq f \right\}$$

→ Def: Let $f: \mathbb{R} \rightarrow \mathbb{R}$ Riemann integrable.

We define the integral $\boxed{\int f}$ of f as the common value

$$\boxed{\sup \left\{ \int \phi : \phi \text{ is a step function and } \phi \leq f \right\}}$$

=

$$\inf \left\{ \int \psi : \psi \text{ is a step function and } \psi \geq f \right\}$$

①

Lecture 28:

Proof of Theorem:

Let $A := \sup \{ \phi : \phi \text{ is a step function, } \phi \leq f \}$,
and $B := \inf \{ \psi : \psi \text{ is a step function, } \psi \geq f \}$.

Since f is Riemann integrable, we know that, $\forall \epsilon > 0$,
there exist step functions ϕ, ψ , with $\phi \leq f \leq \psi$,
st. $\int \psi - \int \phi < \epsilon$. (I.e., we can find elements of A and B arbitrarily close to each other!)

Thus, there certainly exist step functions above and below f , thus A and B are non-empty.

- A is bounded above, in fact by any element of B . Indeed, let ψ step, with $\psi \geq f$.

Then, for any ϕ step, with $\phi \leq f$,
we have $\phi \leq \psi$,
thus $\int \phi \leq \int \psi$.

Therefore, A is bounded above by $\int \psi$. ($\in \mathbb{R}$)

So, A has a least upper bound $\sup A \in \mathbb{R}$,

and $\sup A \leq \underbrace{\int \psi}_{\text{an upper bound of } A}$.

(2)

Since ϕ was an arbitrary step function $\geq f$,
we have that $\sup_{\epsilon \in \mathbb{R}} A_\epsilon$ is a lower bound of B .
So, B has a greatest lower bound $\inf B \in \mathbb{R}$,
and $\sup_{\epsilon \in \mathbb{R}} A_\epsilon \leq \inf B$

α lower bound of B the greatest lower bound of B

Now, suppose that $\sup A \neq \inf B$.

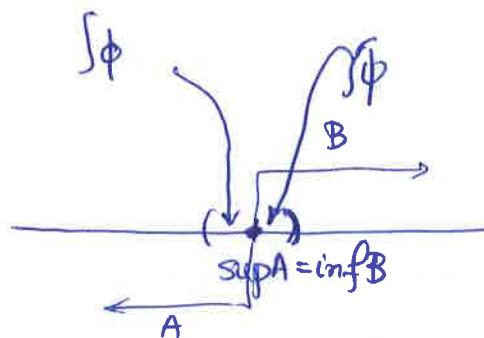
Then, for any ϕ step, $\phi \leq f$,
and any ψ step, $\psi \geq f$,
we have $\int \phi \leq \sup A \neq \inf B \leq \int \psi$,
thus $\int \psi - \int \phi \geq \underbrace{\inf B - \sup A}_{\geq 0}$.

Therefore, the definition of Riemann integrability
of f is not satisfied for $\varepsilon = \inf B - \sup A > 0$,
contradiction.

So, $\sup A = \inf B$.

(3)

\leftarrow Let $\varepsilon > 0$. f bounded $\Rightarrow \sup A, \inf B \in \mathbb{R}$



(why? what step function $\psi \geq f$ can we find?)

Let $\varepsilon > 0$. There exists a step function $\phi \leq f$

$$\text{s.t. } \sup A - \frac{\varepsilon}{2} < \int \phi \leq \sup A$$

because
 $\sup A$ is \geq
any element
of A :

$\int \phi$

is because $\sup A - \frac{\varepsilon}{2}$
not a lower bound
of A

Similarly, there exists a step function $\psi \geq f$

$$\text{s.t. } \inf B \leq \int \psi < \inf B + \frac{\varepsilon}{2}$$

as $\inf B \leq$ any
element of B

as $\inf B + \frac{\varepsilon}{2}$
not a lower
bound of B

$$\text{So: } \sup A - \frac{\varepsilon}{2} < \int \phi \leq \int \psi < \inf B + \frac{\varepsilon}{2}$$

$$\Rightarrow \int \psi - \int \phi < \left(\inf B + \frac{\varepsilon}{2} \right) - \left(\sup A - \frac{\varepsilon}{2} \right) = \varepsilon,$$

for these step functions $\phi \leq f \leq \psi$ So, f R-integrable. ■

4

Very useful for calculating integrals!

\rightarrow Theorem: $f: \mathbb{R} \rightarrow \mathbb{R}$ is Riemann integrable



there exist sequences $(\phi_n)_{n \in \mathbb{N}}$ and $(\psi_n)_{n \in \mathbb{N}}$ of step functions, with

$$\phi_n \leq f \leq \psi_n \quad \forall n \in \mathbb{N},$$

such that $\int \psi_n - \int \phi_n \rightarrow 0$.

*

If $(\phi_n)_{n \in \mathbb{N}}$ and $(\psi_n)_{n \in \mathbb{N}}$ are sequences of step functions satisfying *, then

$$\int \psi_n \rightarrow \int f \quad \text{and} \quad \int \phi_n \rightarrow \int f.$$

Proof: Of (\iff):

(\rightarrow) Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be Riemann integrable.

Then, for every $n \in \mathbb{N}$, it holds that:

there exist step functions ϕ_n, ψ_n , with

$$\phi_n \leq f \leq \psi_n, \quad \text{s.t. } (\exists) \int \psi_n - \int \phi_n < \frac{1}{n}.$$

$\downarrow n \rightarrow +\infty$ $\downarrow n \rightarrow +\infty$
 0 0

By the sandwich lemma, $\int \psi_n - \int \phi_n \rightarrow 0$.

(5)

(\Leftarrow) Let $\epsilon > 0$. There exists $n_0 \in \mathbb{N}$ s.t.

$$\int \psi_n - \int \phi_m < \epsilon \quad \text{if } n \geq n_0.$$

In particular, $\int \psi_{n_0} - \int \phi_{n_0} < \epsilon$, for the step functions $\psi_{n_0} \leq f \leq \phi_{n_0}$.

Since $\epsilon > 0$ was arbitrary, f is Riemann integrable.

Now, let $(\phi_n)_{n \in \mathbb{N}}, (\psi_n)_{n \in \mathbb{N}}$ be sequences of step functions satisfying ④.

Since $\phi_n \leq f \leq \psi_n \quad \forall n \in \mathbb{N}$,

we have, by the definition of $\int f$, that

$$\int \phi_n \leq \int f \leq \int \psi_n, \quad \forall n \in \mathbb{N}$$

Thus: $|\int \phi_n - \int f| = \int f - \int \phi_n \leq \int \psi_n - \int \phi_n \rightarrow 0,$
 $\text{so } \int \phi_n \rightarrow \int f.$

Similarly, $|\int \psi_n - \int f| = \int f - \int \psi_n \leq \int \psi_n - \int \phi_n \rightarrow 0,$
 $\text{so } \int \psi_n \rightarrow \int f.$ ■

(6)

Now, we introduce some notation.

for any $f: \mathbb{R} \rightarrow \mathbb{R}$ bounded, with bounded support in $[a, b]$, and for any partition $\mathcal{P} = \{x_0, x_1, \dots, x_n\}$ of $[a, b]$, with $a = x_0 < x_1 < \dots < x_n = b$, we define

the lower step function $f_*(\mathcal{P})$ of f w.r.t. \mathcal{P} to be the step function

$$f_*(\mathcal{P})(x) = \sum_{i=1}^n m_i \chi_{(x_{i-1}, x_i]}(x) + \sum_{i=0}^n f(x_i) \chi_{\{x_i\}}(x),$$

$\forall x \in \mathbb{R}$,

and the upper step function $f^*(\mathcal{P})$ of f w.r.t. \mathcal{P} to be the step function

$$f^*(\mathcal{P})(x) = \sum_{i=1}^n M_i \chi_{(x_{i-1}, x_i]}(x) + \sum_{i=0}^n f(x_i) \chi_{\{x_i\}}(x),$$

$\forall x \in \mathbb{R}$,

where $m_i = \inf \{f(x) : x \in (x_{i-1}, x_i)\}$,

and $M_i = \sup \{f(x) : x \in (x_{i-1}, x_i)\}$.

The following is a very useful criterion for Riemann-integrability. In particular, (ii) provides an easy way to choose step functions below and above f , that approximate f well (for our purposes).

Prop: Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a bounded function with bounded support in $[a, b]$.

Then, the following are equivalent: (i) and (iii) say exactly the same thing

(i) f is Riemann integrable.

(ii) For all $\epsilon > 0$, there exist $a = x_0 < x_1 < \dots < x_n = b$, such that,

$$\text{if } M_i = \sup \{ f(x) : x \in (x_{i-1}, x_i) \}$$

$$\text{and } m_i = \inf \{ f(x) : x \in (x_{i-1}, x_i) \},$$

$$\text{then } \sum_{i=1}^n (M_i - m_i)(x_i - x_{i-1}) < \epsilon.$$

(iii) For all $\epsilon > 0$, there exist $a = x_0 < x_1 < \dots < x_n = b$, such that

$$\text{actually, again LHS} = \int f^*(P) - \int f(P)$$

for $P = \{x_0, x_1, \dots, x_n\}$

$$\sum_{j=1}^n \sup_{x, y \in (x_{i-1}, x_i)} |f(x) - f(y)| \cdot (x_i - x_{i-1}) < \epsilon.$$

$M_i - m_i$

(8)

Notice that all this Proposition says is the following , which is what we are actually going to prove :

→ Prop * : Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a bounded function, this is not an extra assumption
 with bounded support, contained in $[a, b]$.

Then,

f is Riemann integrable

\iff

$\forall \epsilon > 0$, there exists a partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a, b]$, with $x_0 < x_1 < \dots < x_n$,

$\begin{matrix} \parallel & \parallel \\ a & b \end{matrix}$

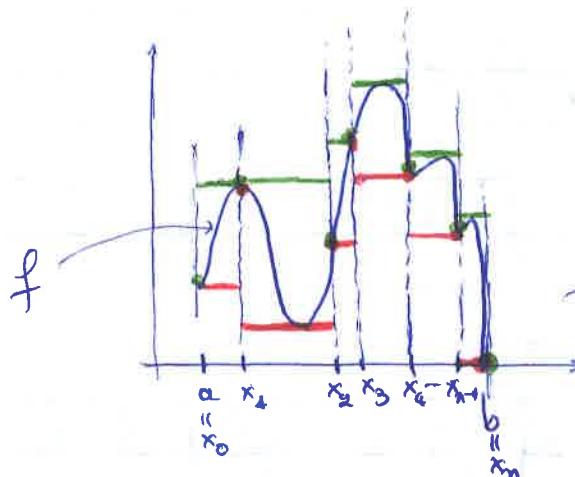
s.t. $\int f^*(P) - \int f_*(P) < \epsilon$.

If f is R-integrable, then certainly it is bounded and has bounded support. We just mention it to specify an interval $[a, b]$ on which we'll work.

(I.e., in the definition of Riemann integrability, we can replace the general step functions with lower and upper step functions of f , w.r.t. the same partition (depending on ϵ)).

Lecture 29:

Δ You should imagine $f^*(P)$ and $f^*(P)$ as the step functions closest to f (below and above it) w.r.t. the partition $P = \{x_0, x_1, \dots, x_n\}$:



Green: $f^*(P)$.

Red: $f_*(P)$

w.r.t. $\{x_0, x_1, \dots, x_n\}$, we cannot find any other _{step} functions $\phi \leq f \leq \psi$ that are closer to f .

Indeed, since

$$f^*(P)(x) = \begin{cases} \sup \{f(x) : x \in (x_{i-1}, x_i)\} & \text{when } x \in (x_{i-1}, x_i), \\ & \forall i=1, \dots, n \\ f(x), & x = x_0, x_1, \dots, x_n \end{cases}$$

$f^*(P) \geq f$, and if ψ step function with $\psi \geq f$, then $\psi \geq f^*(P)$.

And since $f_*(P)(x) = \begin{cases} \inf \{f(x) : x \in (x_{i-1}, x_i)\} & \text{when } x \in (x_{i-1}, x_i), \\ & \forall i=1, \dots, n, \\ f(x), & x = x_0, x_1, \dots, x_n \end{cases}$

$f_*(P) \leq f$, and if ϕ step function with $\phi \leq f$, then $\phi \leq f_*(P)$.

(2)

Proof of Prop^{*}:

(\Leftarrow) Obvious : Let $\varepsilon > 0$. We know that there exists a partition P of $[a,b]$, s.t.

$$\int f^*(P) - \int f_*(P) < \varepsilon.$$

Since $f_*(P) \leq f \leq f^*(P)$, the definition
step functions

of Riemann integrability of f is satisfied for this $\varepsilon > 0$. Since $\varepsilon > 0$ was arbitrary, f is Riemann integrable.

(\Rightarrow) Let $\varepsilon > 0$. Since f is Riemann integrable, there exist step functions ϕ, ψ st.:

$$\phi \leq f \leq \psi \text{ and } \int \psi - \int \phi < \varepsilon.$$

We list in order the potential jump points of both ϕ and ψ :

$x_0 \underset{a}{<} x_1 \underset{||}{<} \dots \underset{||}{<} x_n < x_{n+1} \underset{b}{<} \dots$ Consider the partition
 $P = \{x_0, x_1, \dots, x_n\}$ of $[a,b]$.

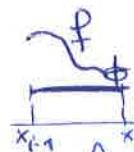
Then, ϕ and ψ are step functions wrt. P , and clearly $\phi \leq f_*(P)$, $\psi \geq f^*(P)$:

(3)

For any (x_{i-1}, x_i) , ϕ and ψ are constant of (x_{i-1}, x_i) . So:

- $\forall x \in (x_{i-1}, x_i), \underbrace{\phi(x)}_{\text{same for all } x \in (x_{i-1}, x_i)} \leq f(x)$

$$\rightarrow \forall x \in (x_{i-1}, x_i), \phi(x) \leq \inf \{f(x) : x \in (x_{i-1}, x_i)\} = f^*(P)(x),$$

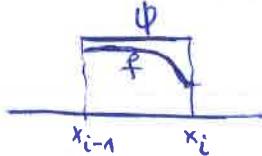


$$\forall i=1, \dots, n.$$

Also, for $x = x_0, x_1, \dots, x_n$: $\phi(x) \leq f(x) = f^*(P)(x)$.

So, $\phi \leq f^*(P)$.

- $\forall x \in (x_{i-1}, x_i), f(x) \leq \underbrace{\psi(x)}_{\text{same for all } x \in (x_{i-1}, x_i)}$ \Rightarrow



$$\rightarrow \underbrace{\sup \{f(x) : x \in (x_{i-1}, x_i)\}}_{f^*(P)(x)} \leq \psi(x), \forall x \in (x_{i-1}, x_i),$$

$$\forall i=1, \dots, n.$$

Also, for $x = x_0, \dots, x_n$: $f(x) \leq \psi(x)$.

$$\underbrace{f^*(P)(x)}_{f^*(P)(x)}$$

So, $f^*(P) \leq \psi$.

Thus, $\phi \leq f^*(P) \leq f \leq f^*(P) \leq \psi$

$$\rightarrow \phi \leq f^*(P) \leq f^*(P) \leq \psi \xrightarrow{\text{all step functions}}$$

$$\rightarrow \int \phi \leq \int f^*(P) \leq \int f^*(P) \leq \int \psi. \text{ So, } \underline{\int f^*(P) - f^*(P)} \leq \int \psi - \int \phi \leq \epsilon. \blacksquare$$

(4)



Prop * means that, instead of working with all possible step functions when deciding integrability and calculating integrals, we can work with a smaller class of step functions, the lower and upper step functions of f w.r.t. all possible partitions of an interval that contains the support of f .

→ Corollary 1: Let f be bounded with bounded support, contained in some closed interval $[a, b]$. Then:

f is Riemann integrable

there exists a sequence $(P_n)_{n \in \mathbb{N}}$ of partitions of $[a, b]$,
st. $\int f^*(P_n) - \int f_*(P_n) \xrightarrow{n \rightarrow \infty} 0$.

In that case, $\int f = \lim_{n \rightarrow \infty} \int f_*(P_n) \left(= \lim_{n \rightarrow \infty} \int f^*(P_n) \right)$

Proof: (\Leftarrow) $(f_*(P_n))_{n \in \mathbb{N}}$ sequence of step functions below f ,

$(f^*(P_n))_{n \in \mathbb{N}}$ sequence of step functions above f)

and $\int f^*(P_n) - \int f_*(P_n) \xrightarrow{n \rightarrow \infty} 0$.

So, f is Riemann integrable, and $\int f = \lim_{n \rightarrow \infty} \int f^*(P_n) = \lim_{n \rightarrow \infty} \int f_*(P_n)$

(5)

\rightarrow f Riemann integrable \Rightarrow

there exist sequences of step functions $(\phi_n)_{n \in \mathbb{N}}$

and $(\psi_n)_{n \in \mathbb{N}}$, with

$$\phi_n \leq f \leq \psi_n \quad \forall n \in \mathbb{N},$$

$$\text{s.t. } \int \psi_n - \int \phi_n \xrightarrow{n \rightarrow \infty} 0.$$

for all $n \in \mathbb{N}$, let P_n be the partition of $[a, b]$ consisting of a, b , and all possible jump points of both ψ_n and ϕ_n . Then,

ϕ_n and ψ_n are step functions w.r.t. P_n ,

$$\text{and } \phi_n \leq f^*(P_n) \leq f \leq f^*(P_n) \leq \psi_n, \quad \forall n \in \mathbb{N}$$

$f^*(P_n)$ and $f^*(P_n)$ approximate f ,
in the best possible way, given P_n .

$$\text{So: } \forall n \in \mathbb{N}, \quad 0 \leq \int f^*(P_n) - \int f_*(P_n) \leq \int \psi_n - \int \phi_n \xrightarrow{n \rightarrow \infty} 0$$

$$\text{By the sandwich lemma, } \int f^*(P_n) - \int f_*(P_n) \xrightarrow{n \rightarrow \infty} 0.$$

Since we have found sequences $(f^*(P_n))_{n \in \mathbb{N}}$ above and below f , with $(f_*(P_n))_{n \in \mathbb{N}}$

(6)

$$\int f^*(P_n) - \int f_*(P_n) \rightarrow 0,$$

we have that $\int f = \lim_{n \rightarrow \infty} \int f_*(P_n)$ ($= \lim_{n \rightarrow \infty} \int f^*(P_n)$).

→ Corollary 2: Let f be bounded, with bounded support, contained in the interval $[a, b]$. Then,

f is Riemann integrable



$$\sup \left\{ \int f_*(P) : \text{all } P \text{ partitions of } [a, b] \right\}$$

"A" =

$$\inf \left\{ \int f^*(P) : \text{all } P \text{ partitions of } [a, b] \right\}$$

"B"

In that case, $\int f = \sup A (= \inf B)$

Proof: (\Leftarrow) $A \subseteq \left\{ \int \phi : \underbrace{\phi}_{A'} \text{ step function, } \phi \leq f \right\}$

$$\rightarrow \sup A \leq \sup A'$$

$B \subseteq \left\{ \int \psi : \underbrace{\psi}_{B'} \text{ step function, } \psi \geq f \right\}$

$$\rightarrow \inf B' \leq \inf B$$

we have seen
that $\sup A' \leq \sup B'$
always

(7)

$$\rightarrow \sup A \leq \sup A' \leq \inf B' \leq \inf B$$

$\sup A = \inf B$ by assumption $\sup A = \boxed{\sup A' = \inf B}$ $\sup A' = \inf B$ $\rightarrow f$ is Riemann integrable,

$$\text{and } \int f = \sup A' = \sup A = \inf B.$$

$$(\Leftarrow) \sup A \leq \sup A' \xrightarrow{f \text{ Riemann integrable}} \int f.$$

$$A \subseteq A' = \{f: \phi \text{ step function, } \phi \leq f\}$$

$$\text{Similarly, } \inf B \geq \int f.$$

We have shown that there exists a sequence (Corollary 4)

$(P_n)_{n \in \mathbb{N}}$ of partitions of $[a, b]$, s.t.

$$\underbrace{\int f_k(P_n)}_{\substack{\in A, \\ k \in \mathbb{N}}} \rightarrow \int f$$

$$\text{and } \underbrace{\int f^*(P_n)}_{\substack{\in B, \\ k \in \mathbb{N}}} \rightarrow \int f.$$

Now, suppose that $\sup A \neq \int f$.

$$\text{Let } \varepsilon = \int f - \sup A \geq 0.$$

Since $\int f_k(P_n) \xrightarrow{n \rightarrow \infty} 0$,

there exists $n_0 \in \mathbb{N}$ s.t. $f_k(P_{n_0}) \in (\sup A, \int f + \varepsilon)$,
 $\int f_k(P_{n_0}) > \sup A$, contradiction.

thus s.t. $\int f_k(P_{n_0}) > \sup A$, contradiction.

$$\text{So, } \sup A = \int f.$$

Similarly, $\inf B = \int f$. $\int \rightarrow \sup A = \inf B = \int f$.

■

→ Δ Notice that, as f Riemann integrable $\Rightarrow f$ bounded with bounded support, we've really been defining definite integrals, We make this more precise:

→ **Def:** Let $f: [a, b] \rightarrow \mathbb{R}$. We say that f is Riemann integrable if $\tilde{f}: \mathbb{R} \rightarrow \mathbb{R}$, with $\tilde{f}(x) = \begin{cases} f(x), & x \in [a, b] \\ 0, & \text{otherwise} \end{cases}$

is Riemann integrable. Then, we define $\int_a^b f(x) dx := \int \tilde{f}$.

• Let $f: \mathbb{R} \rightarrow \mathbb{R}$. We say that f is Riemann integrable on $[a, b]$ if $f|_{[a, b]}$ is Riemann integrable, i.e. if $f|_{[a, b]}$ is R-integrable. Then, $\int_a^b f(x) dx = \int f|_{[a, b]}(x) dx = \int_a^b f(x) dx$.

the restriction of f on $[a, b]$, $f|_{[a, b]}: [a, b] \rightarrow \mathbb{R}$.

⚠ Everything that we have proved or will prove for \int also holds for \int_a^b .

(9)

Basic properties of the Riemann integral:

Let $f, g : \mathbb{R} \rightarrow \mathbb{R}$ be Riemann integrable,
for $[a, b]$

and $\lambda, \mu \in \mathbb{R}$. Then:

(i) $\lambda f + \mu g$ is Riemann integrable, and

$$\int_{\text{(or } \int_a^b)} (\lambda f + \mu g) = \lambda \int f + \mu \int g.$$

(ii) If $f \geq 0$, then $\int f \geq 0$.

If $f \geq g$, then $\int f \geq \int g$.

(iii) $|f|$ is Riemann integrable, and $\left| \int f \right| \leq \int |f|$.

(iv) $f \cdot g$ is Riemann integrable.

Proof: f, g Riemann integrable. Thus,

we can find sequences $(a_n)_{n \in \mathbb{N}}, (b_n)_{n \in \mathbb{N}},$
 $(c_n)_{n \in \mathbb{N}}, (d_n)_{n \in \mathbb{N}}$ of step functions, such
that

(10)

$$a_n \leq f \leq b_n \quad \forall n \in \mathbb{N}$$

and $\int a_n \rightarrow \int f$,
 $\int b_n \rightarrow \int f$

and $c_n \leq g \leq d_n \quad \forall n \in \mathbb{N}$

and $\int c_n \rightarrow \int g$,
 $\int d_n \rightarrow \int g$.

(i) We will prove this for $\lambda, \mu \geq 0$. One works similarly for the other cases.

Since $\lambda \geq 0$, we have

$$\underbrace{\lambda \cdot a_n}_{\text{step function}} \leq \lambda \cdot f \leq \underbrace{\lambda \cdot b_n}_{\text{step function}} \quad \forall n \in \mathbb{N},$$

and since $\mu \geq 0$:

$$\underbrace{\mu \cdot c_n}_{\text{step function}} \leq \mu \cdot g \leq \underbrace{\mu \cdot d_n}_{\text{step function}}, \quad \forall n \in \mathbb{N}$$

$$\Rightarrow \underbrace{\lambda a_n + \mu c_n}_{\text{step function}} \leq \lambda f + \mu g \leq \underbrace{\lambda b_n + \mu d_n}_{\text{step function}}, \quad \forall n \in \mathbb{N}.$$

(14)

$$\text{And: } \int(\lambda b_n + \mu d_n) - \int(\lambda a_n + \mu c_n) =$$

$$= \lambda \cdot \left(\int b_n - \int a_n \right) + \mu \cdot \left(\int d_n - \int c_n \right) \xrightarrow{n \rightarrow \infty} 0.$$

$\downarrow n \rightarrow \infty$ $\downarrow n \rightarrow \infty$

$$\text{So, } \int \lambda f + \mu g = \lim_{n \rightarrow \infty} \int (\lambda b_n + \mu d_n) =$$

$$= \lambda \cdot \lim_{n \rightarrow \infty} \int b_n + \mu \cdot \lim_{n \rightarrow \infty} \int d_n = \lambda \int f + \mu \int g.$$

(ii)

For $f \geq 0$:Since $f \leq b_n \quad \forall n \in \mathbb{N}$, and $f \geq 0$,we have $b_n \geq 0 \quad \forall n \in \mathbb{N}$

$\overbrace{\int b_n}^{\substack{\rightarrow \\ b_n \text{ step} \\ \text{function}}} \geq \int 0 = 0, \quad \forall n \in \mathbb{N}.$

Thus, $\underbrace{\lim_{n \rightarrow \infty} \int b_n}_{\substack{\parallel \\ " \int f "}} \geq 0$.

for $f \geq g$: $f-g \geq 0$, thus $\underbrace{\int(f-g)}_{\substack{\parallel \\ \int f - \int g}} \geq 0 \rightarrow \int f \geq \int g$, by (i)

(iii) • $|f|$ is Riemann integrable:

Let $\epsilon > 0$.

notice that this interval also contains the support of $[a, b]$.

Since f is Riemann integrable, it has some bounded support $[a, b]$. And, for this $\epsilon > 0$, there exist $a = x_0 < x_1 < \dots < x_n = b$, such that

$$\sum_{i=1}^n \sup_{x, y \in (x_{i-1}, x_i)} |f(x) - f(y)| \cdot (x_i - x_{i-1}) < \epsilon.$$

\leq

$$||f(x) - f(y)||,$$

by triangle inequality.

$$\text{So, } \sum_{i=1}^n \sup_{x, y \in (x_{i-1}, x_i)} |f(x) - f(y)| \cdot (x_i - x_{i-1}) < \epsilon.$$

Since $\epsilon > 0$ was arbitrary, $|f|$ is Riemann integrable.

• $|\int f| \leq \int |f| :$

We need to show that

$$-\int |f| \leq \int f \leq \int |f|.$$

This is true by (ii), since $-|f| \leq f \leq |f|$,

and thus

$$\underbrace{-\int |f|}^{\text{if } f \text{ bounded}} \leq \int f \leq \int |f|.$$

- $\int |f|$,
by (i)

- (iv) • first, we show that f^2 is Riemann integrable:
 f R-integrable $\iff f$ bounded, i.e. $\exists N > 0$ s.t. $|f(x)| \leq N \forall x \in \mathbb{R}$.

$$\begin{aligned} |(f(x))^2 - (f(y))^2| &= |f(x) - f(y)| \cdot |f(x) + f(y)| \leq \\ &\leq |f(x) - f(y)| \cdot (|f(x)| + |f(y)|) \leq \\ &\leq 2M \cdot |f(x) - f(y)|, \quad \forall x, y \in \mathbb{R}. \end{aligned}$$

Since f is Riemann integrable, we know that there exist $a = x_0 < x_1 < \dots < x_n = b$ (where $[a, b]$ is a closed interval containing the support of f , and thus of $|f|$), such that

$$\sum_{i=1}^n \sup_{x, y \in (x_{i-1}, x_i)} |f(x) - f(y)| \cdot (x_i - x_{i-1}) < \frac{\epsilon}{2M}.$$

(14)

$$\begin{aligned}
 \text{Thus, } & \sum_{i=1}^n \sup_{x,y \in (x_{i-1}, x_i)} |(f(x))^2 - (f(y))^2| \cdot (x_i - x_{i-1}) \leq \\
 & \leq \sum_{i=1}^n \sup_{x,y \in (x_{i-1}, x_i)} (2M \cdot |f(x) - f(y)|) \cdot (x_i - x_{i-1}) = \\
 & = 2N \cdot \sum_{i=1}^n \sup_{x,y \in (x_{i-1}, x_i)} |f(x) - f(y)| \cdot (x_i - x_{i-1}) < \\
 & < 2M \cdot \frac{\epsilon}{2M} = \epsilon.
 \end{aligned}$$

Since $\epsilon > 0$ was arbitrary, f^2 is Riemann integrable.

- f.g is Riemann integrable:

We notice that $f.g = \frac{1}{4} \cdot ((f+g)^2 - (f-g)^2)$,

which is Riemann integrable:

f, g Riemann integrable $\xrightarrow{(i)}$ $f+g$ and $f-g$

are Riemann integrable $\xrightarrow{} (f+g)^2$ and $(f-g)^2$

are Riemann integrable $\xrightarrow{(i)} \frac{1}{4} \cdot ((f+g)^2 - (f-g)^2)$ Riemann integrable. ■

\rightarrow Thm: Any monotone function $f: [a,b] \rightarrow \mathbb{R}$ is Riemann integrable.

Proof: WLOG, we assume that f is increasing.

for each $n \in \mathbb{N}$, we will find a partition P_n

of $[a,b]$, such that

$$\int_a^b f^*(P_n) - \int_a^b f_*(P_n) \xrightarrow{n \rightarrow \infty} 0. \text{ Then, the proof will be complete.}$$

Let $n \in \mathbb{N}$; we create the partition P_n by splitting $[a,b]$ in n equal parts: $a < a + \frac{b-a}{n} < a + 2 \cdot \frac{b-a}{n} < \dots < a + (n-1) \frac{b-a}{n} < b$,

and we take $P_n := \{ \underbrace{a}_{x_0}, \underbrace{a + \frac{b-a}{n}}_{x_1}, \dots, \underbrace{a + (n-1) \frac{b-a}{n}}_{x_{n-1}}, \underbrace{b}_{x_n} \}$.

In each of the intervals $I_i = \left(\underbrace{a + (i-1) \frac{b-a}{n}}_{x_{i-1}}, \underbrace{a + i \frac{b-a}{n}}_{x_i} \right)$,

we have:

$$\sup_{x \in I_i} f(x) \leq f(x_i)$$

and $f(x_{i-1}) \leq \inf_{x \in I_i} f(x)$

Thus, and
$$\begin{aligned} f^*(P_n) &\leq \sum_{i=1}^n f(x_i) \cdot \chi_{I_i} \\ f_*(P_n) &\geq \sum_{i=1}^n f(x_{i-1}) \cdot \chi_{I_i} \end{aligned} \quad \left. \right\} \Rightarrow$$

(16)

$$\begin{aligned}
 & \rightarrow \int_a^b f^*(P_n) - \int_a^b f_*(P_n) \leq \underline{\hspace{10cm}} \\
 & \leq \sum_{i=1}^n f(x_i) \cdot \underbrace{|I_i|}_{\frac{b-a}{n}} - \sum_{i=1}^n f(x_{i-1}) \cdot \underbrace{|I_i|}_{\frac{b-a}{n}} = \\
 & = \frac{b-a}{n} \cdot \left(\sum_{i=1}^n f(x_i) - \sum_{i=1}^n f(x_{i-1}) \right) = \\
 & = \frac{b-a}{n} \cdot \left(f(x_n) - f(x_0) \right) = \frac{(b-a) \cdot (f(b) - f(a))}{n} \\
 & \xrightarrow[n \rightarrow +\infty]{} 0.
 \end{aligned}$$

So, f is Riemann integrable.

! → Thm: Every continuous $f: [a, b] \rightarrow \mathbb{R}$ is Riemann integrable.

Proof: Let $\epsilon > 0$. $f: \underbrace{[a, b]}_{\text{closed interval}} \rightarrow \mathbb{R}$ is continuous,

thus uniformly continuous. So, for this $\epsilon > 0$,

(17)

there exists $\delta > 0$ st.

if $x, y \in [a, b]$, with $|x-y| < \delta$, then $|f(x) - f(y)| < \frac{\epsilon}{2(b-a)}$

We split $[a, b]$ in smaller intervals, each of length δ : we achieve this, for instance, by considering

$$a < a + \underbrace{\frac{b-a}{n}}_{\substack{x_0 \\ || \\ x_1}} < a + 2 \cdot \frac{b-a}{n} < \dots < a + (n-1) \frac{b-a}{n} < b \quad , \quad x_n$$

for $n \in \mathbb{N}$ st. $\frac{b-a}{n} < \delta$.

In each of the intervals $I_i = (x_{i-1}, x_i)$, we have:

if $x, y \in I_i$, then $|x-y| < \frac{b-a}{n} < \delta$, thus

$$|f(x) - f(y)| < \frac{\epsilon}{2(b-a)}$$

So, $\sup_{x, y \in I_i} |f(x) - f(y)| \leq \frac{\epsilon}{2(b-a)}$.

Therefore, $\sum_{i=1}^n \sup_{x, y \in I_i} |f(x) - f(y)| \cdot |I_i| \leq$
 $\leq \sum_{i=1}^n \frac{\epsilon}{2(b-a)} \cdot |I_i| = \frac{\epsilon}{2(b-a)} \cdot \underbrace{\sum_{i=1}^n |I_i|}_{b-a} = \frac{\epsilon}{2} < \epsilon$.

(18)

Since $\varepsilon > 0$ was arbitrary, f is Riemann-integrable.

(18)

Since $\epsilon > 0$ was arbitrary, f is Riemann-integrable.

①

Lecture 30:
Example:

↓
there are
non-continuous
functions
that are
not Riemann
integrable

Let $f: [0, 1] \rightarrow \mathbb{R}$,

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$$

f is NOT
Riemann integrable.

Indeed, for any $a = x_0 < x_1 < x_2 < \dots < x_n = b$,

we have: $\forall i=1, \dots, n$,

$$\sup_{x \in (x_{i-1}, x_i)} f(x) = 1, \quad \text{as there exists a rational in } (x_{i-1}, x_i),$$

and $\inf_{x \in (x_{i-1}, x_i)} f(x) = 0, \quad \text{as there exists an irrational in } (x_{i-1}, x_i).$

So,

$$\int_0^1 f^*(\{x_0, \dots, x_n\}) = 1, \quad \text{while} \quad \int_0^1 f_*(\{x_0, \dots, x_n\}) = 0.$$

(2)

$$\text{So, } \int_0^1 f^*(\{x_0, \dots, x_n\}) - \int_0^1 f_*(\{x_0, \dots, x_n\}) = 1 - 0 = 1,$$

for this arbitrary partition of $\{x_0, \dots, x_n\}$
of $[0,1]$.

Thus, for $\epsilon = 1/2$, there exists no partition \mathcal{P}
of $[0,1]$ s.t. $\int_0^1 f^*(\mathcal{P}) - \int_0^1 f_*(\mathcal{P}) < \epsilon$.

So, f is not Riemann integrable.

→ Thm: Let $f: [a,b] \rightarrow \mathbb{R}$. Let $a < c < b$.

Then, f Riemann integrable on $[a,b]$



f is Riemann integrable on $[a,c]$ and on $[c,b]$

In that case, $\int_a^b f = \int_a^c f + \int_c^b f$.

(3)

Proof: (\Rightarrow) Let $\epsilon > 0$.

Since $f: [a, b] \rightarrow \mathbb{R}$ is Riemann integrable,
there exist step functions $\phi, \psi: [a, b] \rightarrow \mathbb{R}$,

with $\phi \leq f \leq \psi$

$$\text{and } \int_a^b \psi - \int_a^b \phi < \epsilon.$$

Let $P = \{x_0, x_1, \dots, x_n\}$, with $x_0 \overset{a}{<} x_1 < \dots < \overset{b}{x_n}$
be the set of potential jump points of ϕ and ψ .
Then, ϕ and ψ are step functions w.r.t. P .

Thus, they are both step functions w.r.t. $P \cup \{c\}$.

So, $\phi|_{[a, c]}$, $\psi|_{[a, c]}$ are step functions w.r.t. $(P \cup \{c\}) \cap [a, c]$
and $\phi|_{[c, b]}$, $\psi|_{[c, b]}$ " " " w.r.t. $(P \cup \{c\}) \cap [c, b]$.

And: $\phi|_{[a, c]} \leq f|_{[a, c]} \leq \psi|_{[a, c]},$

$$\text{and } \int_a^c \psi|_{[a, c]} - \int_a^c \phi|_{[a, c]} = \int \psi \chi_{[a, c]} - \int \phi \chi_{[a, c]}$$

(4)

$$\leq \int (\psi - \phi) < \epsilon.$$

$\psi \geq \phi$
on $[a, b]$,
thus on $[a, c]$ too,

$$\text{so } (\psi - \phi) \chi_{[a, c]} \leq \psi - \phi$$

Since $\epsilon > 0$ was
arbitrary,

$f|_{[a, c]}$ is Riemann integrable.

Similarly, $f|_{[c, b]}$ is Riemann integrable,
as, for this arbitrary $\epsilon > 0$,

$$\phi|_{[c, b]} \leq f|_{[c, b]} \leq \psi|_{[c, b]}$$

and $\int_c^b \phi|_{[c, b]} - \int_c^b \psi|_{[c, b]} < \epsilon.$

(\Leftarrow) $f|_{[a, c]}$ Riemann integrable $\rightarrow f \cdot \chi_{[a, c]}$ Riemann integrable.

$f|_{[c, b]}$ Riemann integrable $\Rightarrow f \cdot \chi_{[c, b]}$ Riemann integrable.

So, $f = f \chi_{[a, c]} + f \chi_{[c, b]}$ is Riemann integrable.

And $\int_a^b f = \int_a^b f \chi_{[a, c]} + \int_a^b f \chi_{[c, b]} = \int_a^c f + \int_c^b f.$

(5)

We will now focus our efforts towards understanding the fundamental theorem of calculus. This theorem has to do with the "mean value" of integrable functions $f: [a, b] \rightarrow \mathbb{R}$ (amongst other things) :

→ Def: Let $f: [a, b] \rightarrow \mathbb{R}$ be Riemann integrable.

We define the mean value of f

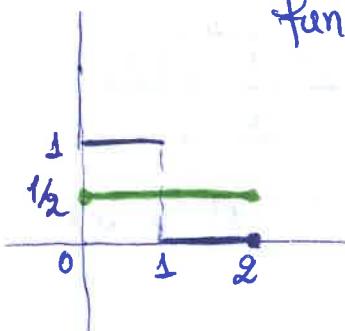
to be the number $\frac{1}{b-a} \cdot \int_a^b f(x)dx$.



Notice that this is the height of the parallelogram with base $[a, b]$ and area $\int_a^b f(x)dx$.

In other words, it is the value of the constant

function with integral on $[a, b]$ equal to $\int_a^b f(x)dx$.



$$\text{mean value } \frac{1}{2}. \left(= \frac{1}{2-0} \cdot \int_0^2 f(x)dx \right)$$

$$1 \cdot (1-0) + 0 \cdot (2-1) = 1.$$

(6)

An observation, to strengthen our intuition that the mean value of f is its "average" value:

Observation: Let $f: [a, b] \rightarrow \mathbb{R}$ be Riemann integrable.

We know that therefore f is bounded, i.e.

$$\sup \{f(x) : x \in [a, b]\} = N \in \mathbb{R}$$

$$\text{and } \inf \{f(x) : x \in [a, b]\} = m \in \mathbb{R}.$$

$$\text{Then: } m \leq \frac{1}{b-a} \int_a^b f(x) dx \leq N.$$

Proof: Consider the step (constant) functions

$$m: [a, b] \rightarrow \mathbb{R}, \quad M: [a, b] \rightarrow \mathbb{R}.$$

$$\text{Then, } m \leq f(x) \leq M, \quad \forall x \in [a, b]$$

$$\Rightarrow \underbrace{\int_a^b m dx}_{m \cdot (b-a)} \leq \int_a^b f(x) dx \leq \underbrace{\int_a^b M dx}_{M \cdot (b-a)}$$

$$\Rightarrow m \leq \frac{1}{b-a} \int_a^b f(x) dx \leq M.$$



(7)

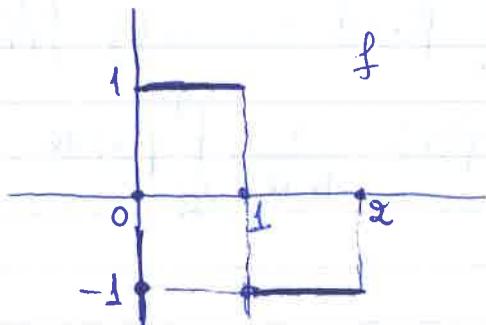
→ **Question:** Let $f: [a, b] \rightarrow \mathbb{R}$ be Riemann integrable.

Is it always true that the mean value of f equals some value of f ?

I.e., are we certain that $\exists c \in [a, b]$, s.t.

$$\frac{1}{b-a} \int_a^b f(x) dx = f(c) ?$$

Answer: Generally NO. Look at this for example:



$$\frac{1}{2-0} \int_0^2 f(x) dx = \frac{1}{2} (1 \cdot 1 + (-1) \cdot 1) = 0,$$

and yet $f(x) \neq 0 \ \forall x \in [0, 2]$.

However, if f is continuous, the answer is YES:

→ **Thm:** Let $f: [a, b] \rightarrow \mathbb{R}$ be continuous.

Then, there exists $c \in [a, b]$ s.t.

$$f(c) = \frac{1}{b-a} \int_a^b f(x) dx.$$

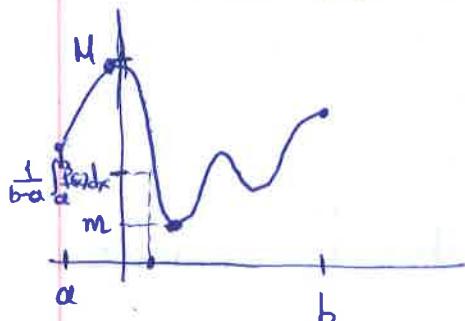
(8)

Proof: f is continuous on the closed interval $[a, b]$.

So, $f([a, b]) = [m, M]$, where

$$m = \min \{ f(x) : x \in [a, b] \} \quad (\in \mathbb{R})$$

$$\text{and } M = \max \{ f(x) : x \in [a, b] \} \quad (\in \mathbb{R})$$



We have shown that

$$m \leq \frac{1}{b-a} \int_a^b f(x) dx \leq M,$$

$$\text{i.e. } \frac{1}{b-a} \int_a^b f(x) dx \in [m, M] = f([a, b])$$

$$\Rightarrow \exists c \in [a, b] \text{ with } f(c) = \frac{1}{b-a} \int_a^b f(x) dx.$$

You can think of this this way if you prefer:

$$\left. \begin{array}{l} \exists x_1 \in [a, b] \text{ s.t. } f(x_1) = m, \\ \exists x_2 \in [a, b] \text{ s.t. } f(x_2) = M, \end{array} \right\}$$

$\overbrace{\text{f cont.}}^{\text{IVT}}$

$$\text{and } m \leq \frac{1}{b-a} \int_a^b f(x) dx \leq M$$

\exists $c \in [a, b]$
 (actually, in closed
 interval with endpoints
 x_1, x_2)

$$\text{s.t. } f(c) = \frac{1}{b-a} \int_a^b f(x) dx$$

amongst other things!

(3)

The fundamental theorem of calculus will imply that the above property not only holds for continuous functions, but also for functions that are derivatives. (and integrable).

→ Theorem : Let $f: [a, b] \rightarrow \mathbb{R}$ be Riemann integrable.

We define $F: [a, b] \rightarrow \mathbb{R}$,

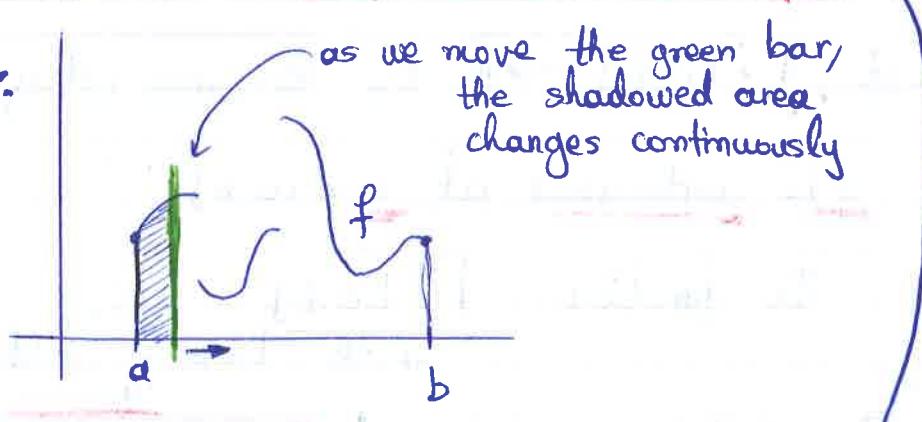
$$\text{with } F(x) = \int_a^x f(t) dt$$

For $x \in [a, b]$,
if $f: [a, x] \rightarrow \mathbb{R}$ is
Riemann integrable, as
 $f: [a, b] \rightarrow \mathbb{R}$ is
Riemann integrable.

Then, F is continuous

(in fact, Lipschitz continuous)

Meaning:



Proof:

$f: [a, b] \rightarrow \mathbb{R}$ is Riemann integrable

$\implies f$ is bounded.

That is, $\exists M > 0$ s.t. $|f(x)| \leq M$, $\forall x \in [a, b]$.

We will show that, for this $M > 0$:

$$|f(x) - f(y)| \leq M \cdot |x - y|, \quad \forall x, y \in [a, b].$$

Indeed, for any two points $x < y$ in $[a, b]$:

$$\begin{aligned} |f(x) - f(y)| &= \left| \int_a^x f(t) dt - \int_a^y f(t) dt \right| = \\ &= \left| - \int_x^y f(t) dt \right| \leq \int_x^y |f(t)| dt \stackrel{\substack{|f(t)| \leq M, \\ t \in [x, y]}}{\leq} \int_x^y M dx = \\ &= M \cdot (y - x) \stackrel{x < y}{=} M \cdot |x - y|. \end{aligned}$$

■

Theorem: Let $f: [a, b] \rightarrow \mathbb{R}$ be Riemann integrable.

If f is continuous at $x_0 \in [a, b]$,

then the function $f: [a, b] \rightarrow \mathbb{R}$,
with $F(x) = \int_a^x f(t) dt \quad \forall x \in [a, b]$,

is differentiable at x_0 , and $F'(x_0) = f(x_0)$.

Another way to think about this: any integrable f is the derivative of something, at its points of continuity.

(11)

Proof: We assume that $x_0 \in (a, b)$
 (and we work similarly for $x_0 = a$ or b).

We will show that $\lim_{h \rightarrow 0} \frac{f(x_0+h) - f(x_0)}{h} = f(x_0)$.

Indeed, let $h > 0$.

$$\begin{aligned}
 \left| \frac{f(x_0+h) - f(x_0)}{h} - f(x_0) \right| &= \left| \frac{\int_a^{x_0+h} f(t) dt - \int_a^{x_0} f(t) dt}{h} - f(x_0) \right| = \\
 &= \left| \frac{\int_{x_0}^{x_0+h} f(t) dt}{h} - f(x_0) \right| \quad \boxed{f(x_0) = \frac{\int_{x_0}^{x_0+h} f(t) dt}{h}} \\
 &= \left| \frac{\int_{x_0}^{x_0+h} f(t) dt - \int_{x_0}^{x_0+h} f(x_0) dt}{h} \right| = \left| \frac{\int_{x_0}^{x_0+h} (f(t) - f(x_0)) dt}{h} \right| \quad \boxed{\text{STANDARD TRICK of changing a number into an integral!}} \\
 &\leq \frac{1}{|h|} \cdot \int_{x_0}^{x_0+h} |f(t) - f(x_0)| dt. \quad \underline{\text{Let } \epsilon > 0.}
 \end{aligned}$$

Since f is continuous at x_0 , there exists $\delta > 0$ s.t.

if $t \in (x_0, x_0 + \delta)$, then $|f(t) - f(x_0)| < \epsilon$.

So, if we consider $0 < h < \delta$, the above calculation gives for any such h that

(12)

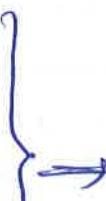
$$\left| \frac{f(x_0+h) - f(x_0)}{h} - f'(x_0) \right| \leq \frac{1}{|h|} \cdot \int_{x_0}^{x_0+h} |f(t) - f(x_0)| dt \stackrel{|h|=h}{\leq}$$

$$< \frac{1}{h} \cdot \int_{x_0}^{x_0+h} \varepsilon dt = \frac{1}{h} \cdot h \cdot \varepsilon = \varepsilon.$$

$x_0 < t < x_0 + h < x_0 + \delta$
 $\Rightarrow |f(t) - f(x_0)| < \varepsilon$

Since $\varepsilon > 0$ was arbitrary,

$$\lim_{h \rightarrow 0^+} \frac{f(x_0+h) - f(x_0)}{h} = f'(x_0)$$



Similarly, we show that $\lim_{h \rightarrow 0^-} \frac{f(x_0+h) - f(x_0)}{h} = f'(x_0)$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{f(x_0+h) - f(x_0)}{h} = f'(x_0).$$

Thus, f is differentiable at x_0 , and $f'(x_0) = f'(x_0)$.

■

This immediately implies the following:

! 1st fundamental theorem of Calculus:

Let $f: [a,b] \rightarrow \mathbb{R}$ be continuous.

Then, the indefinite integral

$$F(x) = \int_a^x f(t) dt \text{ is differentiable on } [a,b]$$

$$\text{and } F'(x) = f(x), \quad \forall x \in [a,b].$$

⚠ This means that the indefinite integral of f is an antiderivative of f .

Indeed, a function $G: [a,b] \rightarrow \mathbb{R}$ is called an antiderivative of $f: [a,b] \rightarrow \mathbb{R}$

$$\text{if } G'(x) = f(x) \quad \forall x \in [a,b].$$

Another way to see this: any continuous $f: [a,b] \rightarrow \mathbb{R}$ is the derivative of something!

different perspective.

Lecture 31:

(1)

→ Lemma: Let $f: [a, b] \rightarrow \mathbb{R}$, with F an antiderivative of f . Then, all the antiderivatives of f are the functions $F+c$, $c \in \mathbb{R}$.

Proof: Let $F, G: [a, b] \rightarrow \mathbb{R}$ both be antiderivatives of f . Then,

$$F'(x) - G'(x) = f(x) - f(x) = 0 \quad \forall x \in [a, b],$$

$$\text{thus } G(x) = F(x) + c \quad \forall x \in [a, b],$$

for some $c \in \mathbb{R}$.



for this we are using that:

If $F: [a, b] \rightarrow \mathbb{R}$ and $F'(x) = 0 \quad \forall x \in [a, b]$,
then $F(x) = c$, for some $c \in \mathbb{R}$.

Proof: Let $x \in (a, b)$. Then, by MVT

on $[a, x]$, we have that

$$\begin{aligned} f(x) - f(a) &= f'(\tilde{x}_x)(x-a) \quad \text{for some } \tilde{x}_x \in (a, x) \\ &= 0 \cdot (x-a) = 0, \end{aligned}$$

thus $f(x) = f(a)$, $\forall x \in [a, b]$.

(2)

Also, for any $c \in \mathbb{R}$, the function $G: [a, b] \rightarrow \mathbb{R}$,

$$G(x) = f(x) + c \quad \forall x \in [a, b],$$

is an antiderivative of f :

$$G'(x) = F'(x) + (c)' = f'(x) + 0 = f'(x) = f(x), \quad \forall x \in [a, b].$$

\rightarrow really, the 1st fundamental Thm of Calculus says this,
 plus that the set of antiderivatives of f is $\neq \emptyset$!

\rightarrow Prop: Let $f: [a, b] \rightarrow \mathbb{R}$ be continuous.

Then,
$$G(x) - G(a) = \int_a^x f(t) dt, \quad \forall x \in [a, b],$$

for any $G: [a, b] \rightarrow \mathbb{R}$ antiderivative of f .
 \hookrightarrow i.e., $G'(x) = f(x) \quad \forall x \in [a, b]$.

In particular,
$$G(b) - G(a) = \int_a^b f(t) dt.$$

Proof: Let $G: [a, b] \rightarrow \mathbb{R}$ be an antiderivative of f .
 Since $\int_a^x f(t) dt, x \in [a, b]$, is an antiderivative of f , we
 have that $G(x) = \int_a^x f(t) dt + c$.

for all $x \in [a, b]$,
 for some $c \in \mathbb{R}$.

In particular,
$$G(a) = \int_a^a f(t) dt + c = 0 + c = c,$$

i.e. the constant c in question is $c = G(a)$.

So,
$$G(x) - G(a) = \int_a^x f(t) dt, \quad \forall x \in [a, b].$$

(3)

⚠ By the above, $\frac{1}{b-a} \int_a^b f(t) dt = \frac{G(b) - G(a)}{b-a} =$
 the mean value of f

MVT for
 $G: [a, b] \rightarrow \mathbb{R}$
 diff.

$$G'(c) = f(c), \text{ for some } c \in (a, b).$$

That is, the mean value of f is a value of f , when f is continuous. This is something we already knew.

→ Example: Let $f: [0, 1] \rightarrow \mathbb{R}$,
 $f(x) = x^2, \forall x \in [0, 1]$.

f is continuous. And we see that

$G: [0, 1] \rightarrow \mathbb{R}$,
 with $G(x) = \frac{x^3}{3},$ is an antiderivative of f :

$$G'(x) = \left(\frac{x^3}{3}\right)' = \frac{3x^2}{3} = x^2 = f(x), \quad \forall x \in [0, 1].$$

By the fundamental theorem of Calculus,

$$\int_0^x f(t) dt = G(x) - G(0) \quad \forall x \in [0, 1]. \text{ I.e. :}$$

$$\int_0^x t^2 dt = \frac{x^3}{3} - \frac{0^3}{3} = \frac{x^3}{3} \quad \forall x \in [0, 1].$$

In particular, $\int_0^1 t^2 dt = \frac{1}{3}$.

The 1st fundamental theorem of Calculus generalises
to any $f: [a,b] \rightarrow \mathbb{R}$ that has an antiderivative.

⚠ $f: [a,b] \rightarrow \mathbb{R}$ continuous always has an antiderivative; we've shown this. But not every Riemann integrable $f: [a,b] \rightarrow \mathbb{R}$ has to have an antiderivative.

2nd fundamental Theorem of Calculus:

Let $f: [a,b] \rightarrow \mathbb{R}$ be Riemann integrable.

Suppose that f has an antiderivative,

$G: [a,b] \rightarrow \mathbb{R}$. Then:

$$\int_a^b f(t) dt = G(b) - G(a).$$



Notice that the above implies that



$$\int_a^x f(t) dt = G(x) - G(a)$$

$t \in [a,b]$, as $f: [a,x] \rightarrow \mathbb{R}$ is Riemann integrable and has antiderivative $G: [a,x] \rightarrow \mathbb{R}$, $t \in [a,b]$.

So, for $f: [a,b] \rightarrow \mathbb{R}$ R-integrable and with an antiderivative,

$\int_a^x f(t) dt$, $x \in [a,b]$ is differentiable, and $(\int_a^x f(t) dt)' = f(x)$, $x \in [a,b]$.

(5)



In literature you may find this theorem written instead as follows:

Let $G: [a, b] \rightarrow \mathbb{R}$ be differentiable. If G' is Riemann integrable on $[a, b]$, then

$$G(b) - G(a) = \int_a^b G'(t) dt.$$

Notice that there is absolutely no difference between the two formulations.

(5)

See how we have deduced that the 1st and 2nd fundamental theorems of Calculus have the same conclusion: that, whether $f: [a, b] \rightarrow \mathbb{R}$

is continuous, OR Riemann integrable with an antiderivative,

then $F: [a, b] \rightarrow \mathbb{R}$,

$$\text{with } F(x) = \int_a^x f(t) dt,$$

is differentiable on $[a, b]$, and

$$\underline{F'(x) = f(x), \quad \forall x \in [a, b]}.$$

(this in turn implies that all the antiderivatives G of f are the functions $G = f + c$, $c \in \mathbb{R}$).

And this implies that, for any antiderivative G of f ,

$$c = G(a) - \cancel{f(a)} = G(a),$$

$$\text{thus } G(b) - G(a) = \int_a^b f(t) dt.$$

There is a difference of perspective though. When f is continuous, the news is that there exists an antiderivative. When we know that f has an antiderivative, the focus is on how to calculate it. That is why the two statements focus on different things.

(7)

And, of course, both theorems tell us how to calculate the definite integral $\int_a^b f(t)dt$: we find an antiderivative G of f on $[a,b]$, and evaluate $G(b) - G(a)$.

Proof of the 2nd Fundamental Theorem of Calculus:

We will show that

$$\int_a^b f^*(P) \leq G(b) - G(a) \leq \int_a^b f^*(P),$$

for any partition $P = \{a = x_0 < x_1 < \dots < x_n = b\}$

of $[a,b]$. Then, since $f: [a,b] \rightarrow \mathbb{R}$ is

Riemann integrable,

$$\int_a^b f(t)dt = \sup \left\{ \int_a^b f^*(P) : P \text{ partitions of } [a,b] \right\} \leq G(b) - G(a),$$

$$\int_a^b f(t)dt = \inf \left\{ \int_a^b f^*(P) : P \text{ partitions of } [a,b] \right\} \geq G(b) - G(a),$$

(8)

thus $\int_a^b f(t) dt = G(b) - G(a).$

So, it suffices to prove $\textcircled{*}$:

Let $P = \{a = x_0 < x_1 < \dots < x_n\}$ be a partition of $[a, b]$.

for all $i=1, 2, \dots, n$,

$$G(x_i) - G(x_{i-1}) = G'(y_i) \cdot (x_i - x_{i-1}) = f(y_i) \cdot (x_i - x_{i-1}),$$

for some $y_i \in (x_{i-1}, x_i)$.

Thus, since $m_i \leq f(y_i) \leq M_i$,
 $\inf \{f(x) : x \in (x_{i-1}, x_i)\} \leq f(y_i) \leq \sup \{f(x) : x \in (x_{i-1}, x_i)\}$

essentially, we have expressed
 $G(x_i) - G(x_{i-1})$ as an integral;
the size of a parallelogram.

we have

$$\sum_{i=1}^n m_i \cdot (x_i - x_{i-1}) \leq \sum_{i=1}^n f(y_i) (x_i - x_{i-1}) \leq \sum_{i=1}^n M_i (x_i - x_{i-1}),$$

$\underbrace{\sum_{i=1}^n m_i \cdot (x_i - x_{i-1})}_{\int_a^b f_*(P)}$ $\underbrace{\sum_{i=1}^n f(y_i) (x_i - x_{i-1})}_{G(x_n) - G(x_0)}$ $\underbrace{\sum_{i=1}^n M_i (x_i - x_{i-1})}_{\int_a^b f^*(P)}$

i.e. $\int_a^b f_*(P) \leq \underbrace{G(x_n) - G(x_0)}_{G(b) - G(a)} \leq \int_a^b f^*(P).$

Thus, $\textcircled{*}$ holds. ■

⚠ Let $f: [a, b] \rightarrow \mathbb{R}$ be Riemann integrable, with an antiderivative $G: [a, b] \rightarrow \mathbb{R}$.

We know now by the 2nd fundamental theorem of calculus that

$$\int_a^x f(t) dt = G(x) - G(a), \quad \forall x \in [a, b].$$

- First of all, $\int_a^b f(t) dt = G(b) - G(a) \Rightarrow \frac{\int_a^b f(t) dt}{b-a} = \frac{G(b) - G(a)}{b-a} =$

$$= \frac{G'(z)}{b-a} = \frac{f(z)}{b-a}, \quad \text{by the mean value theorem for } G: [a, b] \rightarrow \mathbb{R}, \quad \text{for some } z \in (a, b).$$

Thus, even if f is not continuous,

as long as it is the derivative of something,

then its mean value is attained as a value of f . This is really a special case of the fact that the intermediate value theorem holds for derivatives (thus, it holds for $G' = f$).

- More importantly, $\forall x \in (a, b],$

$$\frac{\int_a^x f(t) dt}{x-a} = \frac{G(x) - G(a)}{x-a}$$

the mean value of f on $[a, x]$

(10)

By shrinking $[a, x]$ more and more towards $\{a\}$,

we have:

$$\lim_{x \rightarrow a^+} \frac{\int_a^x f(t) dt}{x-a} = \lim_{x \rightarrow a^+} \frac{G(x) - G(a)}{x-a} = G'(a) = f(a).$$

the averages
of f on intervals
that shrink towards
 $\{a\}$

Thus, the averages of f on intervals that shrink to $\{a\}$ tend to $f(a)$.

This is a special case of the Lebesgue differentiation theorem (look it up if you are interested). Roughly, this says that, for very general functions $f: \mathbb{R}^n \rightarrow \mathbb{R}$, the value $f(a)$ of f at "any" $a \in \mathbb{R}^n$ can be approximated as well as we want by averages of f on tiny balls around a .